

Modeling of fractional dynamics using Lévy walks - recent advances

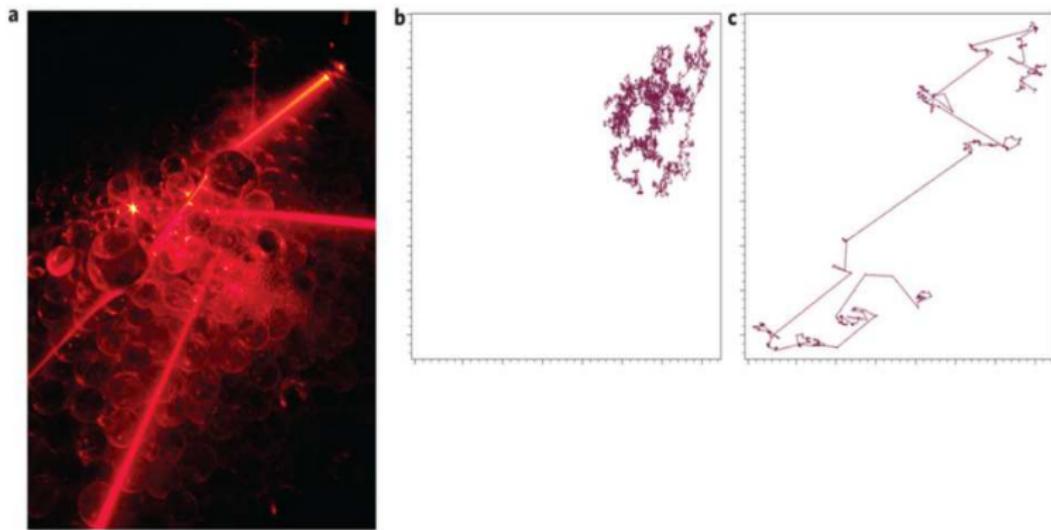
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Fractional PDEs: Theory, Algorithms and Applications
ICERM, Brown University, June 2018

- Examples of applications of Lévy walks
- Basic definitions of Lévy walks
- Asymptotic (diffusion) limits of multidimensional Lévy walks
- Corresponding fractional diffusion equations
- Explicit densities in multidimensional case
- Other results

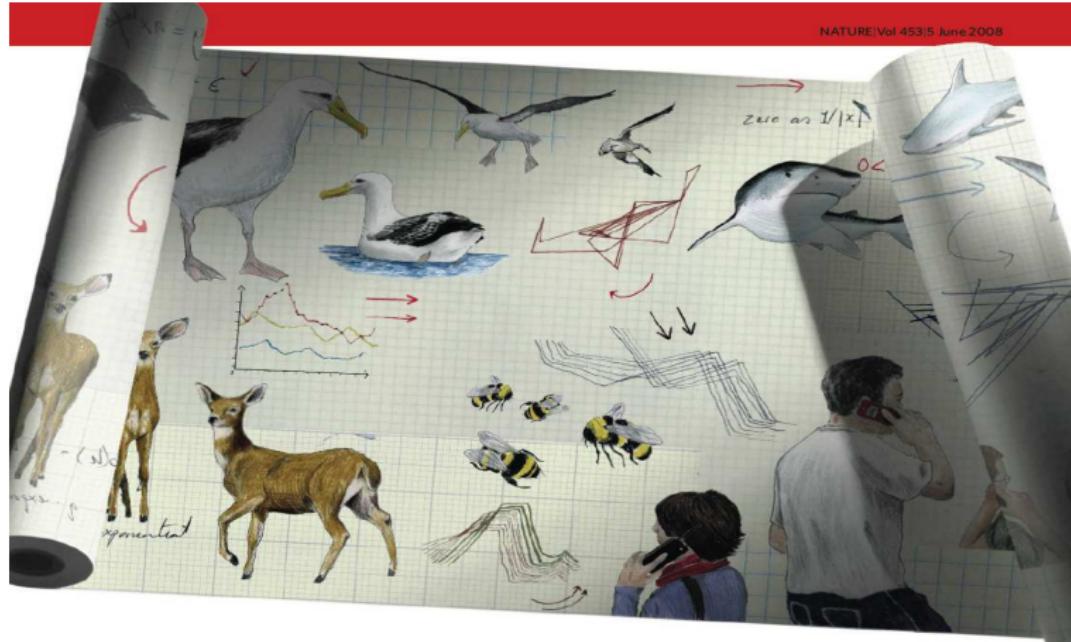
Examples of applications of Lévy walks



Light transport in optical materials

P. Barthelemy, P.J. Bertolotti, D.S. Wiersma, Nature 453, 495 (2008).

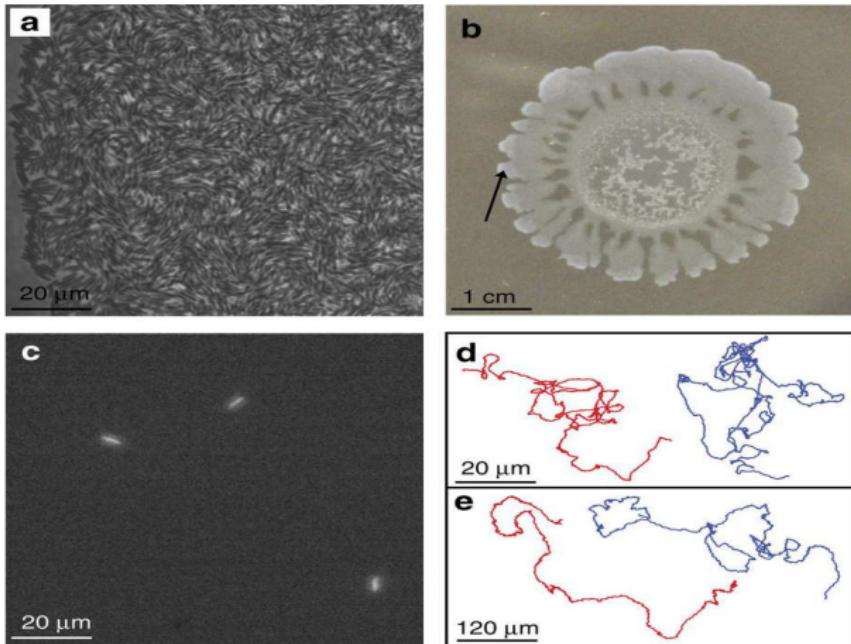
Examples of applications of Lévy walks



Foraging patterns of animals

M. Buchanan, Nature 453, 714 (2008).

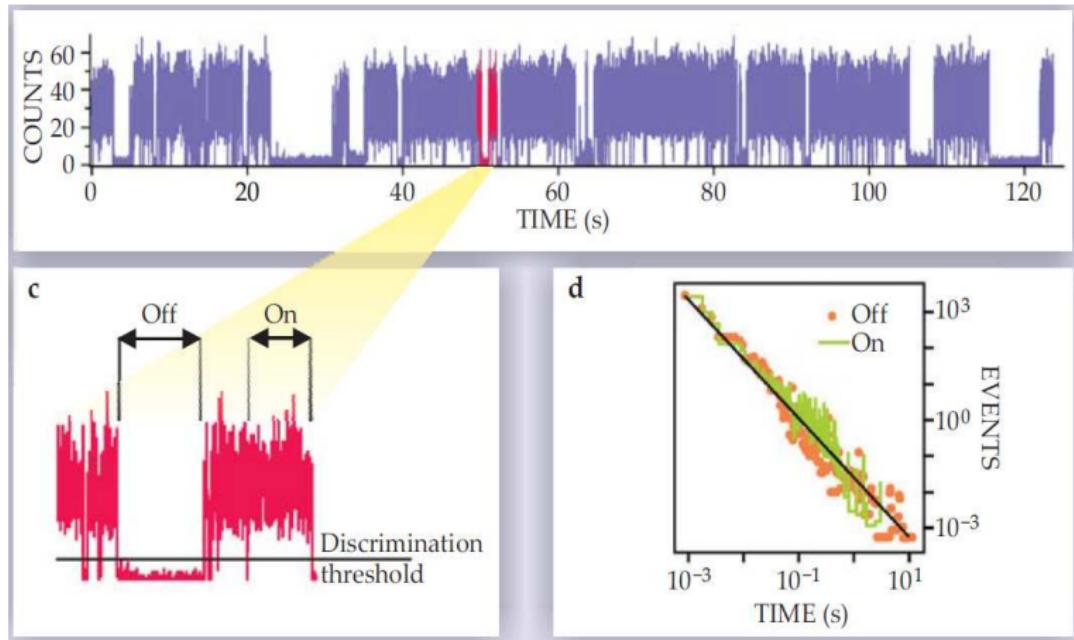
Examples of applications of Lévy walks



Migration of swarming bacteria

G. Ariel et al., Nature Communications (2015).

Examples of applications of Lévy walks

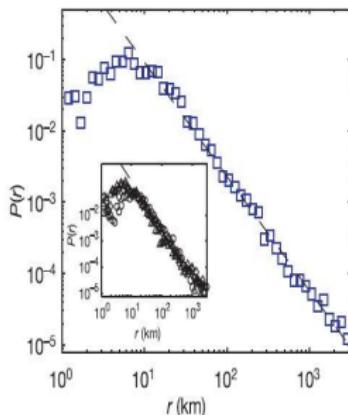
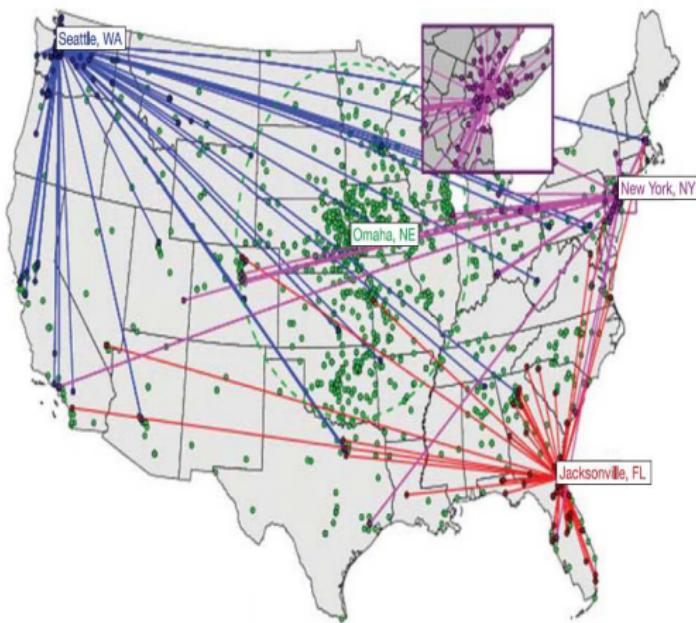


Blinking nanocrystals

G. Margolin, E. Barkai, Phys. Rev. Lett. 94, 080601 (2005)

F.D. Stefani, J.P. Hoogenboom, E. Barkai, Physics Today, 62 (2009)

Examples of applications of Lévy walks



Human travel

D. Brockmann, L. Hufnagel, and T. Geisel, Nature 439, 462 (2006).

- **Waiting times:** $T_i, i = 1, 2, \dots$ – sequence of iid positive random variables with power-law distribution $\mathbb{P}(T_i > t) \propto \frac{1}{t^\alpha}$, $\alpha \in (0, 1)$.

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- **Jumps:**

$$J_i = \Lambda_i T_i$$

where Λ_i are iid random variables with

$$\mathbb{P}(\Lambda_i = 1) = p, \quad \mathbb{P}(\Lambda_i = -1) = 1 - p.$$

They govern the **direction of the jumps** (velocity $v = 1$). $|T_i| = |J_i|$.

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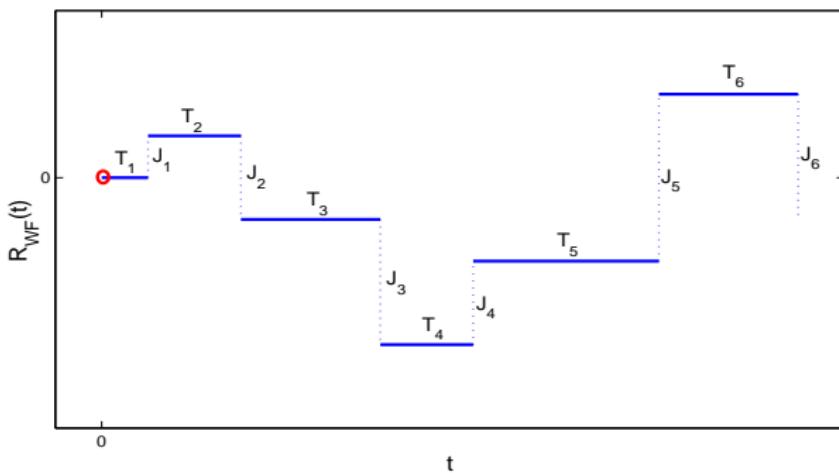
- **Number of jumps up to time t:**

$$N_t = \max\{n \geq 0 : T_1 + \dots + T_n \leq t\}.$$

Definition: Wait-First Lévy Walk – 1D case

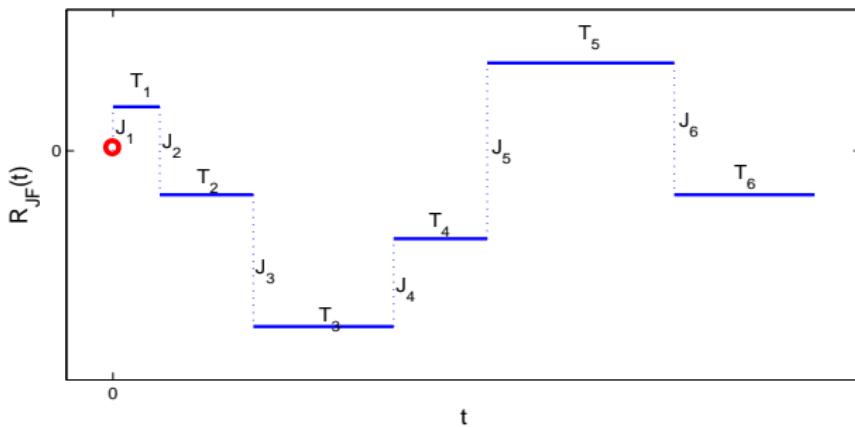
$$R_{WF}(t) = \sum_{i=1}^{N_t} J_i$$

Note that $|R_{WF}(t)| \leq t$.



Definition: Jump-First Lévy Walk – 1D case

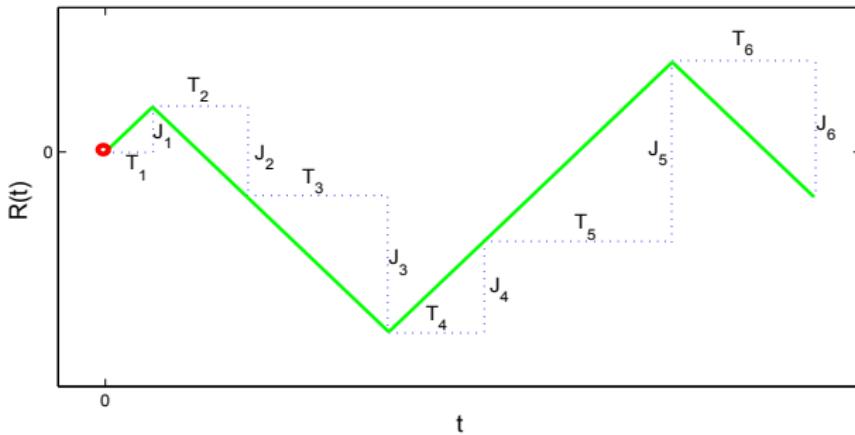
$$R_{JF}(t) = \sum_{i=1}^{N_t+1} J_i$$



Definition: Standard Lévy Walk – 1D case

$$R(t) = \sum_{i=1}^{N_t} J_i + (t - T(N_t)) \Lambda_{N_t+1},$$

where $T(n) = \sum_{i=1}^n T_i$. Note that $|R(t)| \leq t$.



- **Waiting times:** $T_i, i = 1, 2, \dots$ – sequence of iid positive random variables with power-law distribution $\mathbb{P}(T_i > t) \propto \frac{1}{t^\alpha}$, $\alpha \in (0, 1)$.

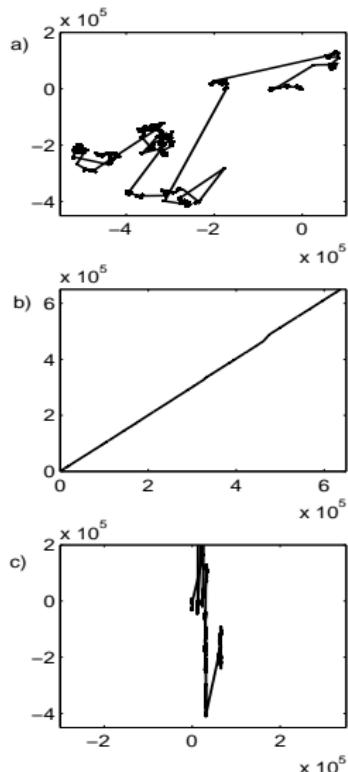
- **Waiting times:** $T_i, i = 1, 2, \dots$ – sequence of iid positive random variables with power-law distribution $\mathbb{P}(T_i > t) \propto \frac{1}{t^\alpha}$, $\alpha \in (0, 1)$.
- **Jumps in \mathbb{R}^d :**

$$J_i = \Lambda_i T_i$$

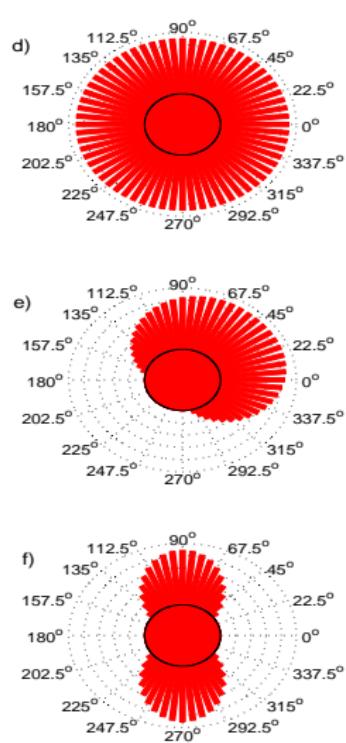
where Λ_i are iid **unit random vectors** in \mathbb{R}^d with the distribution $\Lambda(dx)$ on d -dimensional sphere \mathbb{S}^d . They govern the **direction of the jumps** in \mathbb{R}^d (velocity $v = 1$). We have $|T_i| = \|J_i\|$.

Basic definitions – d -dimensional case

Trajectory



Distribution Λ on \mathbb{S}^2



Wait-First Lévy Walk in \mathbb{R}^d

$$R_{WF}(t) = \sum_{i=1}^{N_t} J_i$$

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Standard Lévy Walk in \mathbb{R}^d

$$R(t) = \sum_{i=1}^{N_t} J_i + (t - T(N_t))\Lambda_{N_t+1},$$

where $T(n) = \sum_{i=1}^n T_i$.

Theorem (Diffusion limit of Wait-First Lévy walk)

The following convergence in distribution holds as $n \rightarrow \infty$

$$\frac{R_{WF}(nt)}{n} \xrightarrow{d} L_\alpha^-(S_\alpha^{-1}(t)).$$

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Here:

- $L_\alpha(t)$ – d -dimensional α -stable Lévy motion (limit of jumps) with Fourier transform

$$\Phi_{L_\alpha(t)}(k) = \exp \left(t \int_{\mathbb{S}^d} |\langle k, s \rangle|^\alpha (i \operatorname{sgn}(\langle k, s \rangle) \tan(\pi \alpha/2) - 1) \Lambda(ds) \right)$$

- $\Lambda(ds)$ - distribution of jump direction

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- Coupling! $|\Delta L_\alpha(t)| = \Delta S_\alpha(t)$

Diffusion limits of Lévy walks – 1D case

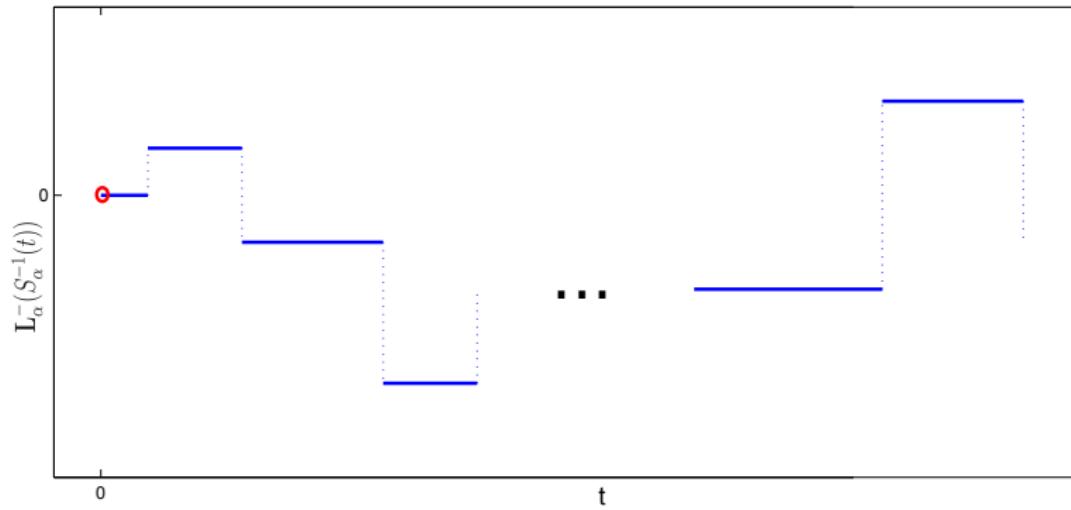


Figure: Trajectory of the diffusion limit of Wait-First Lévy walk. It can have **infinitely many jumps** on finite interval.

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- $\Lambda(ds)$ - distribution of jump direction
- $S_\alpha(t)$ – α -stable subordinator (limit of waiting times), coupling as before.

Diffusion limits of Lévy walks – 1D case

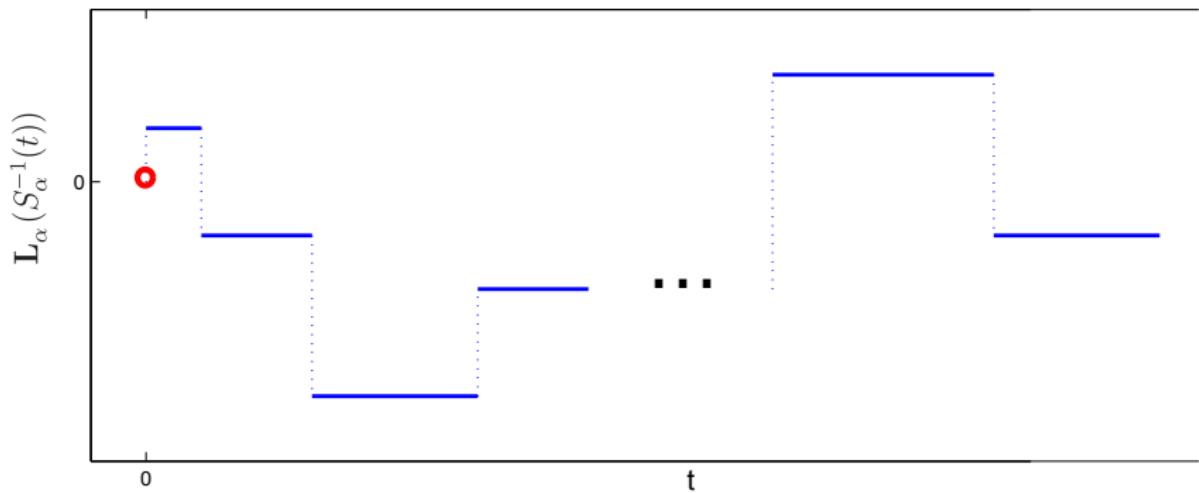


Figure: Trajectory of the diffusion limit of Jump-First Lévy walk. It can have **infinitely many jumps** on finite interval.

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Here:

$$Z(t) = \begin{cases} L_\alpha^-(S_\alpha^{-1}(t)) & \text{if } t \in \mathcal{R} \\ L_\alpha^-(S_\alpha^{-1}(t)) + \frac{t - G(t)}{H(t) - G(t)}(L_\alpha(S_\alpha^{-1}(t)) - L_\alpha^-(S_\alpha^{-1}(t))) & \text{if } t \notin \mathcal{R}, \end{cases}$$

$$\mathcal{R} = \{S_\alpha(t) : t \geq 0\},$$

$$G(t) = S_\alpha^-(S_\alpha^{-1}(t)),$$

$$H(t) = S_\alpha(S_\alpha^{-1}(t))).$$

Diffusion limits of Lévy walks – 1D case

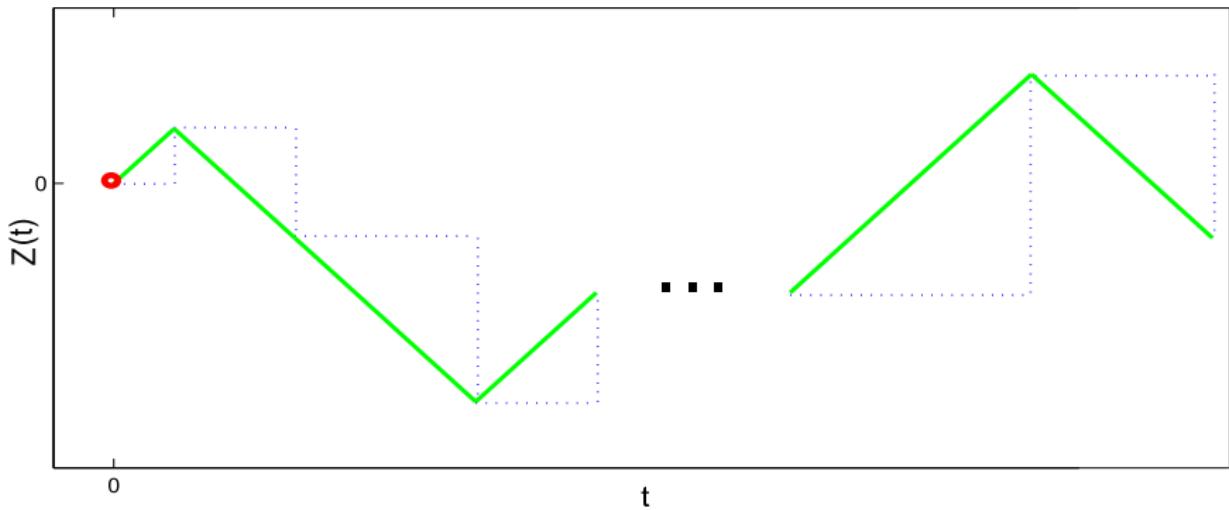


Figure: Trajectory of the diffusion limit of standard Lévy walk. It can have **infinitely many** changes of direction on finite interval.

Fractional material derivative (d -dimensional)

$$\mathbb{D}^{\alpha, \Lambda} p(x, t) = \int_{u \in \mathbb{S}^d} \left(\frac{\partial}{\partial t} + \langle \nabla, u \rangle \right)^\alpha p(x, t) \Lambda(\mathrm{d}u),$$

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In Fourier-Laplace space

$$\mathcal{F}_x \mathcal{L}_t \{ \mathbb{D}^{\alpha, \Lambda} p(x, t) \} = \int_{u \in \mathbb{S}^d} (s - i \langle k, u \rangle)^\alpha \Lambda(\mathrm{d}u) p(k, s).$$

- Wait-First Lévy Walk in \mathbb{R}^d

$$\mathbb{D}^{\alpha, \Lambda} p_{WF}(x, t) = \delta(x) \frac{t^{-\alpha}}{\Gamma(1 - \alpha)},$$

$p_{WF}(x, t)$ - PDF of diffusion limit.

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$$\mathbb{D}^{\alpha, \Lambda} p_{JF}(x, t) = \nu(dx, (t, \infty)),$$

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- Standard Lévy Walk in \mathbb{R}^d

$$\mathbb{D}^{\alpha, \Lambda} p(x, t) = \delta(\|x\| - t) \frac{t^{-\alpha}}{\Gamma(1 - \alpha)},$$

$p(x, t)$ - PDF of diffusion limit.

- **I Method** [D. Froemberg, M. Schmiedeberg, E. Barkai, V. Zaburdaev, Phys. Rev. E 91, 022131 (2015)]

Inversion formula of Fourier-Laplace transform for **1D ballistic processes** using Sokhotsky-Weierstrass theorem.

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- **II Method** – Markov approach

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- We have

$$(L_\alpha^-(S_\alpha^{-1}(t)) = dx, t - G(t-) = dv) = \\ = \nu_{(L_\alpha, S_\alpha)}(\mathbb{R} \times [v, \infty)) U(dx, t - dv) \mathbf{1}_{\{0 \leq v \leq t\}},$$

ν - Lévy measure, U - potential measure.

PDF of Wait-First Lévy Walk(i) if $x \in (-t, 0)$, then

$$p_{WF}(x, t) = \frac{p \sin(\pi\alpha) t^{1-\alpha}}{\pi |x|^{1-\alpha}} \times \frac{(1 - \frac{x}{t})^\alpha}{p^2(1 - \frac{x}{t})^{2\alpha} + (1 - p)^2(1 + \frac{x}{t})^{2\alpha} + 2p(1 - p)(1 - \frac{x}{t})^\alpha(1 + \frac{x}{t})^\alpha \cos(\pi\alpha)},$$

(ii) if $x \in (0, t)$, then

$$p_{WF}(x, t) = \frac{(1 - p) \sin(\pi\alpha) t^{1-\alpha}}{\pi |x|^{1-\alpha}} \times \frac{(1 + \frac{x}{t})^\alpha}{p^2(1 + \frac{x}{t})^{2\alpha} + (1 - p)^2(1 - \frac{x}{t})^{2\alpha} + 2p(1 - p)(1 + \frac{x}{t})^\alpha(1 - \frac{x}{t})^\alpha \cos(\pi\alpha)},$$

(iii) if $|x| \geq t$ then $p_{WF}(x, t) = 0$.

PDFs of Lévy Walks – 1-dimensional case

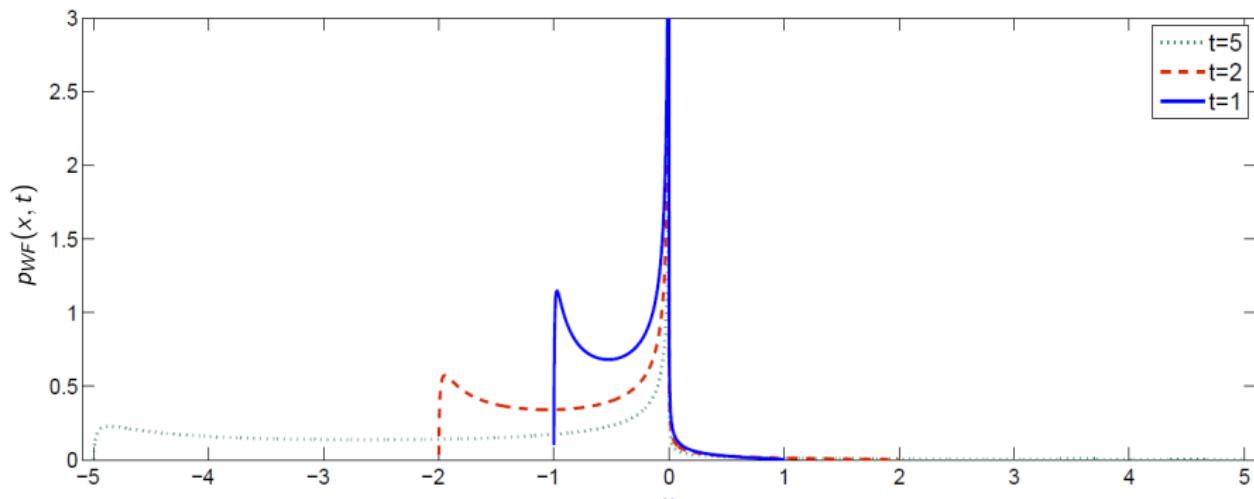


Figure: PDF $p_{WF}(x, t)$ calculated for $\alpha = 0.5$, $p = 0.1$ and different t .

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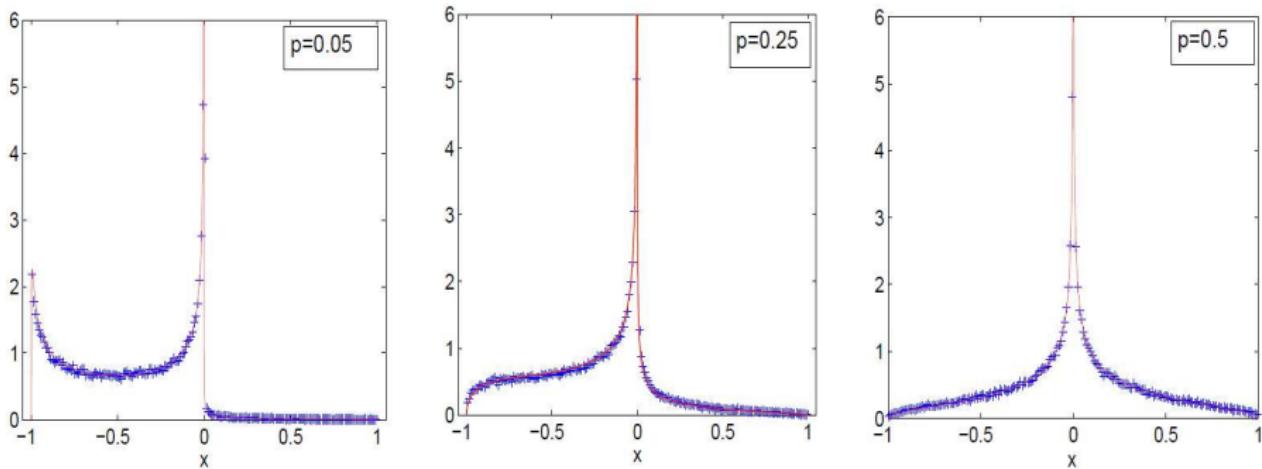


Figure: PDF $p_{WF}(x, t)$ calculated for $\alpha = 0.5$, $t = 1$ and different p .

PDFs of Lévy Walks – 1-dimensional case

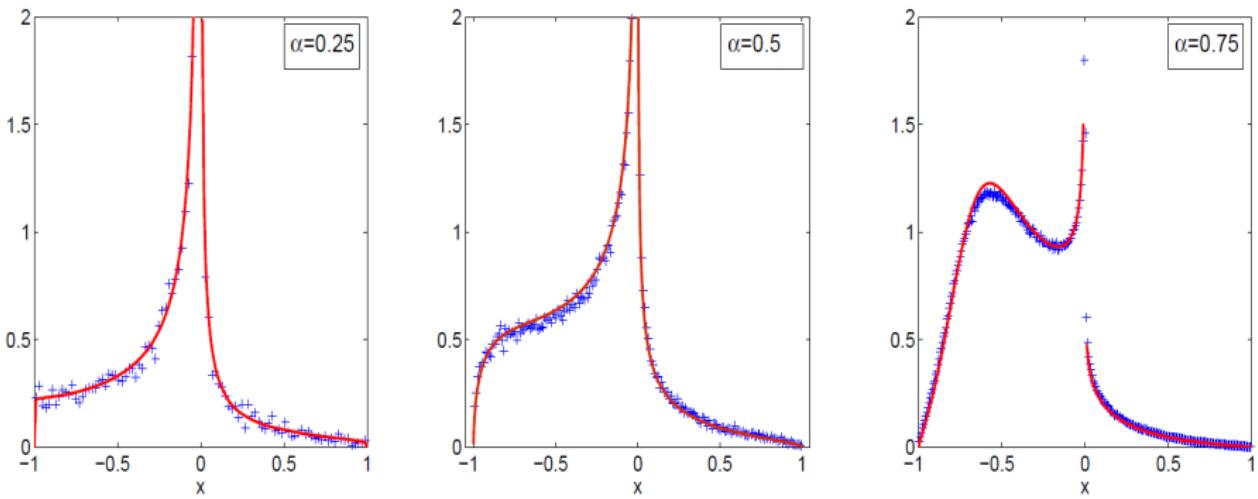


Figure: PDF $p_{WF}(x, t)$ calculated for $p = 0.25$, $t = 1$ and different α .

PDF of Jump-First Lévy Walk(i) if $x < -t$, then

$$p_{JF}(x, t) = \frac{(p-1)\sin(\pi\alpha)}{\pi x} \frac{1}{(1-p)(-x/t-1)^\alpha + p(-x/t+1)^\alpha},$$

(ii) if $x \in [-t, t]$, then

$$p_{JF}(x, t) = \frac{p(1-p)\sin(\pi\alpha)}{\pi x} \times \frac{(1+\frac{x}{t})^\alpha - (1-\frac{x}{t})^\alpha}{p^2(1-\frac{x}{t})^{2\alpha} + (1-p)^2(1+\frac{x}{t})^{2\alpha} + 2p(1-p)(1-\frac{x}{t})^\alpha(1+\frac{x}{t})^\alpha \cos(\pi\alpha)},$$

(iii) if $x > t$, then

$$p_{JF}(x, t) = \frac{p\sin(\pi\alpha)}{\pi x} \frac{1}{p(x/t-1)^\alpha + (1-p)(x/t+1)^\alpha}. \quad (1)$$

PDFs of Lévy Walks – 1-dimensional case

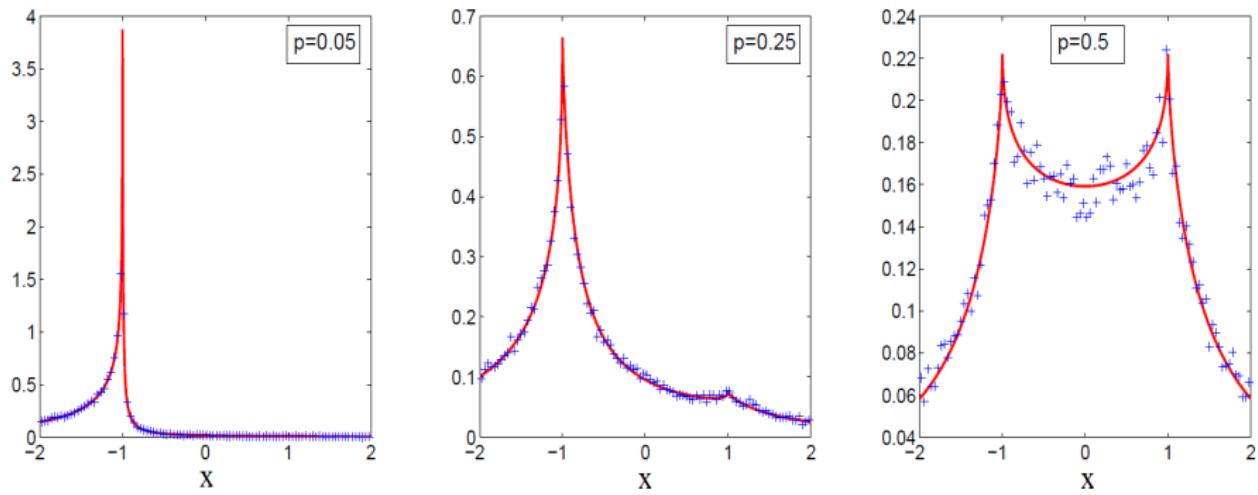


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PDFs of Lévy Walks – 1-dimensional case

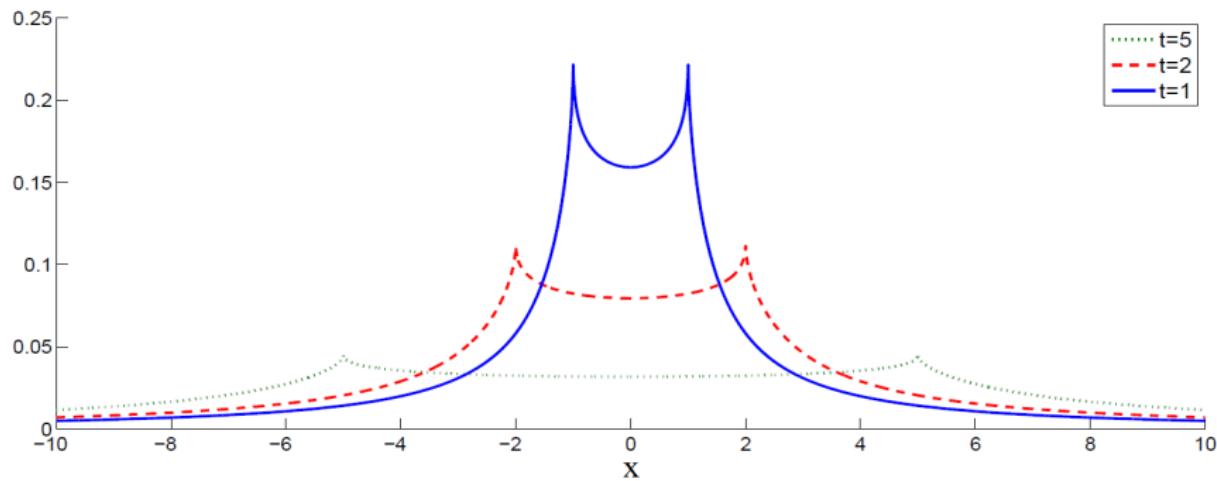


Figure: PDF $p_{JF}(x, t)$ calculated for $\alpha = 0.5$, $p = 0.5$ and different t .

PDF of Standard Lévy Walk

(i) if $|x| < t$, then

$$p(x, t) = \frac{p(1-p) \sin(\pi\alpha)}{\pi t} \times \frac{(1 - \frac{x}{t})^{\alpha-1} (1 + \frac{x}{t})^\alpha + (1 + \frac{x}{t})^{\alpha-1} (1 - \frac{x}{t})^\alpha}{p^2(1 - \frac{x}{t})^{2\alpha} + (1-p)^2(1 + \frac{x}{t})^{2\alpha} + 2p(1-p)(1 - \frac{x}{t})^\alpha(1 + \frac{x}{t})^\alpha \cos(\pi\alpha)}$$

(ii) if $|x| \geq t$, then $p(x, t) = 0$.

PDFs of Lévy Walks – 1-dimensional case

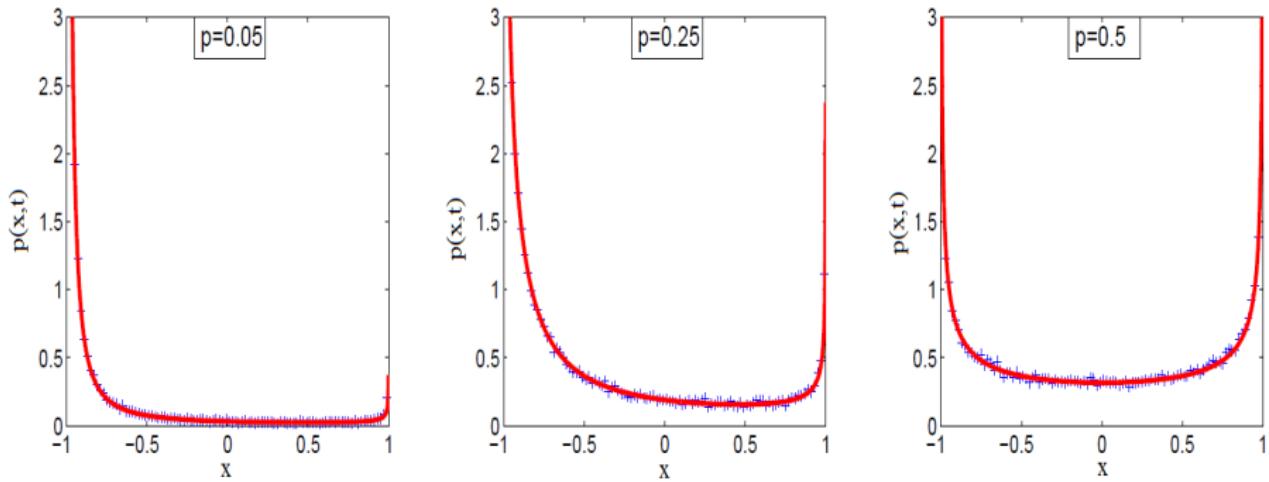


Figure: PDF $p(x, t)$ calculated for $\alpha = 0.5$, $t = 1$ and different p .

PDF of isotropic standard d -dimensional Lévy Walk**General method:**

Let $p(x, t)$, $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, be the PDF of Lévy walk $Z(t) = (Z_1(t), Z_2(t), \dots, Z_d(t))$. The Fourier-Laplace transform of $p(x, t)$ is given by

$$p(k, s) = \frac{1}{s} \frac{\int_{\mathbb{S}^d} \left(1 - \left\langle \frac{ik}{s}, u \right\rangle\right)^{\alpha-1} \Lambda(du)}{\int_{\mathbb{S}^d} \left(1 - \left\langle \frac{ik}{s}, u \right\rangle\right)^\alpha \Lambda(du)},$$

Denote by $\Phi_1(x)$ the PDF of $Z_1(1)$.

(i) **Odd number of dimensions** $d = 2n + 3$.

- We have

$$\Phi_1(x) = -\frac{1}{\pi|x|} \operatorname{Im} \frac{{}_2F_1((1-\alpha)/2, 1-\alpha/2; 3/2+n; \frac{1}{x^2})}{{}_2F_1(-\alpha/2, (1-\alpha)/2; 3/2+n; \frac{1}{x^2})},$$

where ${}_2F_1(a, b; c; x)$ is the hypergeometric function.

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- Using the fact that

$$Z(1) \stackrel{d}{=} \|Z(1)\| V,$$

we get that the PDF $\Phi_R(\cdot)$ of $\|Z(1)\|$ equals

$$\Phi_R(\sqrt{r}) = \frac{2\sqrt{\pi}}{\Gamma(n+3/2)} r^{n+1} (-1)^{n+1} \frac{d^{n+1}}{dr^{n+1}} \Phi_1(\sqrt{r}).$$

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where ${}_2F_1(a, b; c; x)$ is the hypergeometric function.

- Using the fact that

$$Z(1) \stackrel{d}{=} \|Z(1)\| V,$$

we get that the PDF $\Phi_R(\cdot)$ of $\|Z(1)\|$ equals

$$\Phi_R(\sqrt{r}) = \frac{2\sqrt{\pi}}{\Gamma(n+3/2)} r^{n+1} (-1)^{n+1} \frac{d^{n+1}}{dr^{n+1}} \Phi_1(\sqrt{r}).$$

- Finally

$$p(x, t) = \frac{\Gamma(n+3/2)}{2\pi^{n+3/2} t \|x\|^{2n+2}} \Phi_R \left(\frac{\|x\|}{t} \right).$$

PDFs of Lévy Walks – d -dimensional case

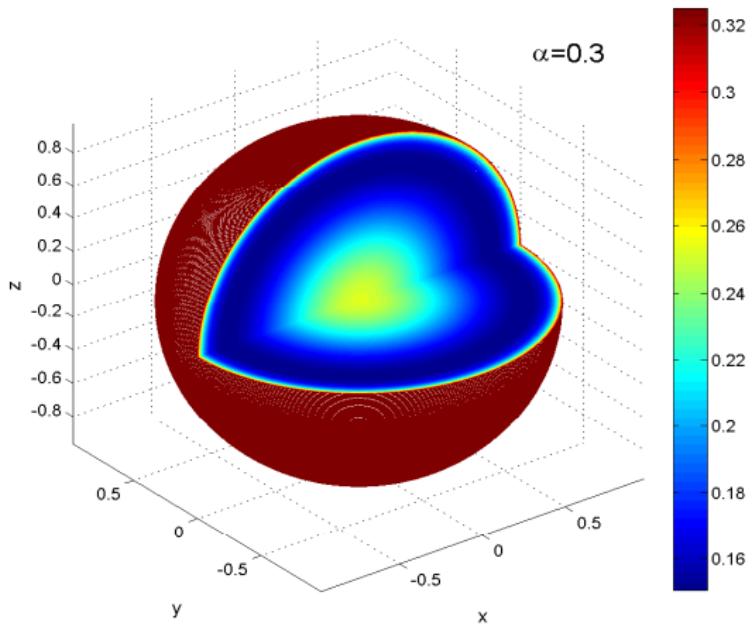


Figure: 3-dimensional PDF $p(x, t)$ calculated for $\alpha = 0.3$ and $t = 1$.

(ii) **Even number of dimensions** $d = 2n + 2$.

- We have

$$\Phi_1(x) = -\frac{1}{\pi|x|} \operatorname{Im} \frac{{}_2F_1((1-\alpha)/2, 1-\alpha/2; 1+n; \frac{1}{x^2})}{{}_2F_1(-\alpha/2, (1-\alpha)/2; 1+n; \frac{1}{x^2})},$$

where ${}_2F_1(a, b; c; x)$ is the hypergeometric function.

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- Moreover

$$\Phi_R(\sqrt{r}) = \frac{2\sqrt{\pi}}{\Gamma(n+1)} r^{n+1/2} D_-^{n+1/2} \{\Phi_1(\sqrt{t})\}(r).$$

Here $D_-^{n+1/2}$ is the right-side Riemann-Liouville fractional derivative of order $n + 1/2$.

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PDFs of Lévy Walks – d -dimensional case

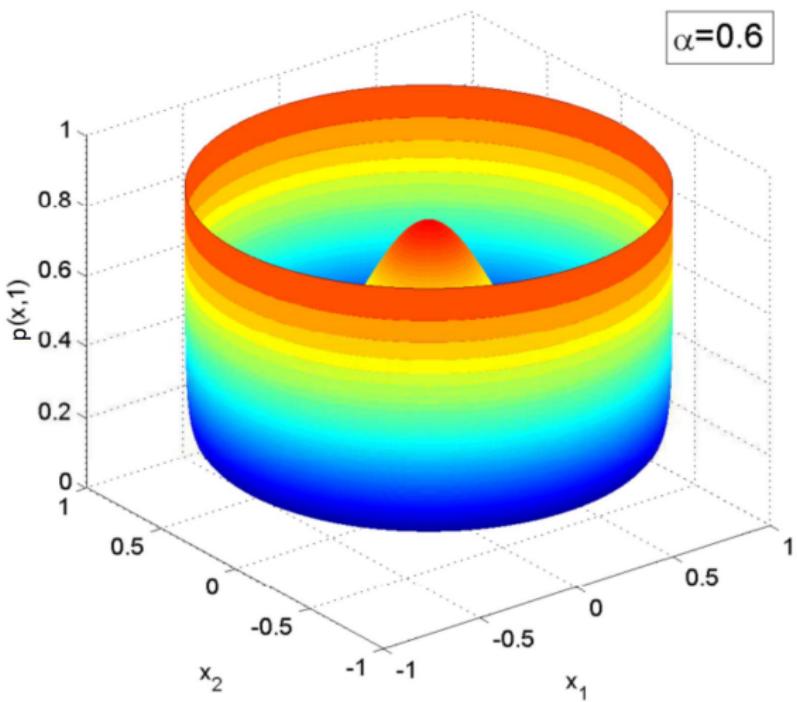


Figure: 2-dimensional PDF $p(x, t)$ calculated for $\alpha = 0.6$ and $t = 1$.

PDF of isotropic Wait-First d -dimensional Lévy Walk

PDF of isotropic Wait-First d -dimensional Lévy Walk

The Fourier-Laplace transform of $p_{WF}(x, t)$ is given by

$$p_{WF}(k, s) = \frac{1}{s} \frac{1}{\int_{\mathbb{S}^d} \left(1 - \left\langle \frac{ik}{s}, u \right\rangle\right)^\alpha \Lambda(du)}.$$

PDF of isotropic Wait-First d -dimensional Lévy Walk

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(i) **Odd number of dimensions** $d = 2n + 3$.

- We have

$$\Phi_1(x) = -\frac{\Gamma(n)}{\Gamma(n+1/2)|x|} \operatorname{Im} \frac{1}{_2F_1(-\alpha/2, (1-\alpha)/2; 3+n/2; \frac{1}{x^2})},$$

$$\Phi_R(\sqrt{r}) = \frac{2\sqrt{\pi}}{\Gamma(n+3/2)} r^{n+1} (-1)^{n+1} \frac{d^{n+1}}{dr^{n+1}} \Phi_1(\sqrt{r}),$$

$$p_{WF}(x, t) = \frac{\Gamma(n+3/2)}{2\pi^{n+3/2} t \|x\|^{2n+2}} \Phi_R \left(\frac{\|x\|}{t} \right).$$

PDF of isotropic Wait-First d -dimensional Lévy Walk**(ii) Even number of dimensions $d = 2n + 2$.**

- We have

$$\Phi_1(x) = -\frac{\Gamma(n)}{\Gamma(n+1/2)|x|} \operatorname{Im} \frac{1}{_2F_1(-\alpha/2, (1-\alpha)/2; 1+n; \frac{1}{x^2})},$$

$$\Phi_R(\sqrt{r}) = \frac{2\sqrt{\pi}}{\Gamma(n+1)} r^{n+1/2} D_-^{n+1/2} \{\Phi_1(\sqrt{t})\}(r),$$

$$p_{WF}(x, t) = \frac{\Gamma(n+1)}{2\pi^{n+1} t \|\mathbf{x}\|^{2n+1}} \Phi_R \left(\frac{\|\mathbf{x}\|}{t} \right).$$

PDF of isotropic Jump-First d -dimensional Lévy Walk

PDF of isotropic Jump-First d -dimensional Lévy Walk

The Fourier-Laplace transform of $p_{JF}(x, t)$ is given by

$$p_{JF}(k, s) = \frac{1}{s} \left(1 - \frac{\left\| \frac{ik}{s} \right\|^\alpha}{\int_{\mathbb{S}^d} \left(1 - \left\langle \frac{ik}{s}, u \right\rangle \right)^\alpha \Lambda(du)} \right).$$

PDF of isotropic Jump-First d -dimensional Lévy Walk

The Fourier-Laplace transform of $p_{JF}(x, t)$ is given by

$$p_{JF}(k, s) = \frac{1}{s} \left(1 - \frac{\left\| \frac{ik}{s} \right\|^\alpha}{\int_{\mathbb{S}^d} \left(1 - \left\langle \frac{ik}{s}, u \right\rangle \right)^\alpha \Lambda(du)} \right).$$

(i) **Odd number of dimensions** $d = 2n + 3$.

- We have

$$\Phi_1(x) = -\frac{\Gamma(n)}{\Gamma(n+1/2) |x|^{\alpha+1}} \operatorname{Im} \frac{\cos(\pi\alpha) + i \sin(\pi\alpha)}{{}_2F_1(-\alpha/2, (1-\alpha)/2; 3/2+n; \frac{1}{x^2})},$$

$$\Phi_R(\sqrt{r}) = \frac{2\sqrt{\pi}}{\Gamma(n+3/2)} r^{n+1} (-1)^{n+1} \frac{d^{n+1}}{dr^{n+1}} \Phi_1(\sqrt{r}),$$

$$p_{JF}(x, t) = \frac{\Gamma(n+3/2)}{2\pi^{n+3/2} t \|x\|^{2n+2}} \Phi_R \left(\frac{\|x\|}{t} \right).$$

PDF of isotropic Jump-First d -dimensional Lévy Walk**(ii) Even number of dimensions $d = 2n + 2$.**

- We have

$$\Phi_1(x) = -\frac{\Gamma(n)}{\Gamma(n+1/2)|x|^{\alpha+1}} \operatorname{Im} \frac{\cos(\pi\alpha) + i\sin(\pi\alpha)}{{}_2F_1(-\alpha/2, (1-\alpha)/2; 1+n; \frac{1}{x^2})},$$

$$\Phi_R(\sqrt{r}) = \frac{2\sqrt{\pi}}{\Gamma(n+1)} r^{n+1/2} D_-^{n+1/2} \{\Phi_1(\sqrt{t})\}(r),$$

$$p_{JF}(x, t) = \frac{\Gamma(n+1)}{2\pi^{n+1} t \|x\|^{2n+1}} \Phi_R \left(\frac{\|x\|}{t} \right).$$

- diffusion limits and governing equations for non-ballistic Lévy walks
- path properties of Lévy walks (martingale properties, upper and lower limits, variation etc.)
- distributed order Lévy walks
- Lévy walks and flights in quenched disorder
- multipoint PDFs of Lévy walks
- aging Lévy walks
- ergodic properties of Lévy walks

Other results – multipoint PDFs of Lévy walks

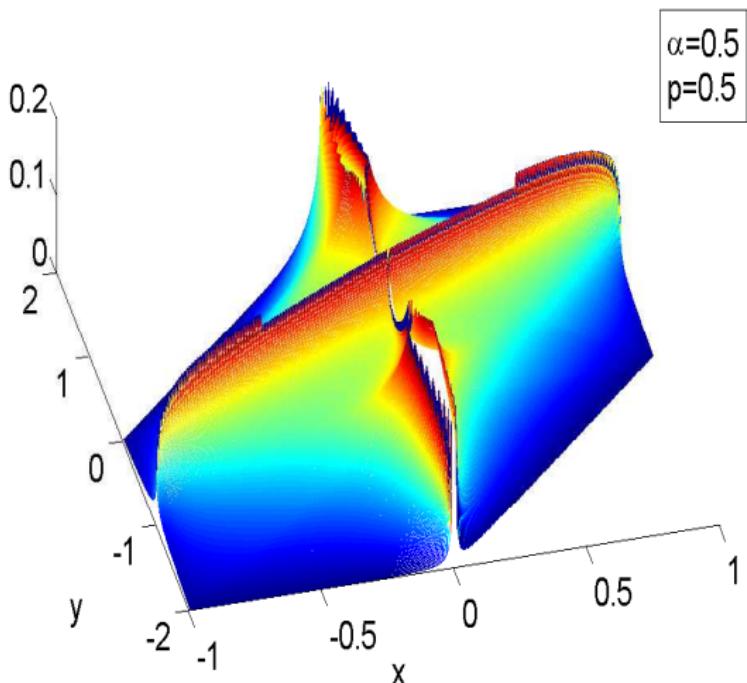


Figure: PDF of $(Z(t_1), Z(t_2))$ for $\alpha = 0.5$, $p = 0.5$, $t_1 = 1$, $t_2 = 2$.

Other results – aging Lévy walks

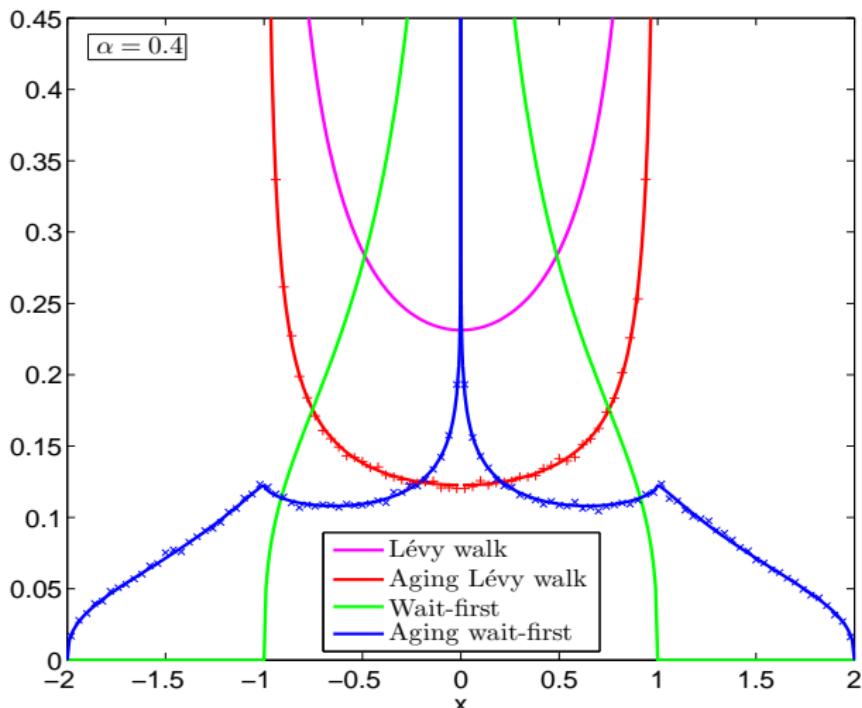


Figure: PDFs of standard and aging Lévy walk.

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